Horner's Method to find a root approximately

Principles of Horner's Method

① If $f(p)$ and $f(q)$ are of opposite signs, then $f(x)=0$ has a root between $x=p$ and $x=q$

② Transformation of an equation by diminishing its roots by a constant

③ Transformation of an equation by multiplying its roots by 10.

Method

Suppose one root of $f(x)=0$ is $\alpha \cdot \beta\gamma\delta \ldots$ with its integral part α and decimal part $\beta\gamma\delta\ldots$ Then $\beta, \gamma, \delta \ldots$ are integers from 0, to 9.

To find α: Find the consecutive integers p and $p+1$ such that $f(p)$ and $f(p+1)$ are of opposite signs. Then $\alpha = p$

To find β: Diminish the roots by α and then multiply the roots by 10. Then the integral part of the root which is one of $0, 1, \ldots 9$ is β.

To find γ: Diminish the roots by β and then multiply the roots by 10. to get γ as an integral part.

To find δ: Repeat the process. ~~again~~. and so on.

Example: 1 By using Horner's method show that the equation $x^3 - 3x + 1 = 0$ has a root between 1 and 2, find it to three decimal places.

Proof:

We are given that the required root is lies between 1 and 2.

$\therefore$ It is of the form $1.\beta\gamma\delta$

We shall diminish the roots ~~of~~ by the integral part of the root (namely 1)

$$f(x) = x^3 + 0x^2 - 3x + 1$$

$$\begin{array}{c|cccc}
1 & 1 & 0 & -3 & 1 \\
 & 0 & 1 & 1 & -2 \\
\hline
 & 1 & 1 & -2 & \boxed{-1} \\
 & 0 & 1 & 2 & \\
\hline
 & 1 & 2 & \boxed{0} & \\
 & 0 & 1 & & \\
\hline
 & 1 & \boxed{3} & & \\
 & 0 & & & \\
\hline
 & 1 & & &
\end{array}$$

Now the transformed equation is:

$$x^3 + 3x^2 + 0x - 1 = 0 \longrightarrow (1)$$

We shall multiply the roots of this equation by 10

Transformed equation is:

$$(1) \Rightarrow \quad 1(x^3) + 10(3x^2) + 100(0x) + 1000(-1) = 0$$

$$f(x) = x^3 + 30x^2 + 0x - 1000 = 0 \longrightarrow (2)$$

Since by substituting $x = 0, 1, 2, 3, 4, 5, 6 \ldots$ in equation (2)

~~we~~ we see that

when $x = 0$; (2) $\Rightarrow f(0) = -1000$ (negative)

when $x = 1$; (2) $\Rightarrow f(1) = 1 + 30 + 0 - 1000 = -969$ (negative)

when $x = 2$; (2) $\Rightarrow f(2) = 8 + 120 + 0 - 1000 = -872$ (negative)

when $x = 3$; (2) $\Rightarrow f(3) = 27 + 270 + 0 - 1000 = -703$ (negative) when $x=4$; ② ⟹ $f(4) = 64 + 480 + (-1000) = -456$ (negative)

when $x=5$; ② ⟹ $f(5) = 125 + 750 + (-1000) = -125$ (negative)

when $x=6$; ② ⟹ $f(6) = 216 + 1080 + (-1000) = 296$ (Positive)

Here $f(5)$ is negative and $f(6)$ is Positive

∴ A root of ② is lies between 5 and 6

∴ we shall diminish the roots of the equation ② by 5.

$$
\begin{array}{c|cccc}
5 & 1 & 30 & 0 & -1000 \\
 & 0 & 5 & 175 & 875 \\
\hline
 & 1 & 35 & 175 & \boxed{-125} \\
 & 0 & 5 & 200 & \\
\hline
 & 1 & 40 & \boxed{375} & \\
 & 0 & 5 & & \\
\hline
 & 1 & \boxed{45} & & \\
 & 0 & & & \\
\hline
 & \boxed{1} & & &
\end{array}
$$

The transformed equation is :

$$\cancel{f(x)=}\ x^3 + 45x^2 + 375x - 125 = 0$$

we shall multiply the roots of this equation by 10.

the the transformed equation becomes,

$$g(x) = 1(x^3) + 10(45x^2) + 100(375x) + 1000(-125) = 0$$

$$x^3 + 450x^2 + 37500x - 125000 = 0 \longrightarrow ③$$

A root of ③ is lies between 3 and 4.

∴ $f(3)$ = negative
$f(4)$ = Positive
(or)
$\dfrac{\text{Constant term}}{\text{Coeff of } x} = \dfrac{125000}{37500} = 3.33$
A root is lies between 3 and 4

So we shall diminish the roos by 3

$$
\begin{array}{c|cccc}
3 & 1 & 450 & 37500 & -125000 \\
 & 0 & 3 & 1359 & 116577 \\
\hline
 & 1 & 453 & 38859 & \boxed{-8423} \\
 & 0 & 3 & 1368 & \\
\hline
 & 1 & 456 & \boxed{40227} & \\
 & 0 & 3 & & \\
\hline
 & 1 & \boxed{459} & & \\
 & 0 & & & \\
\hline
 & \boxed{1} & & &
\end{array}
$$

Transformed equation : $x^3 + 459x^2 + 40227x - 8423 = 0$

we shall multiply the roots of this equation by 10, then the transformed equation is,

$$1(x^3) + 10(459)x^2 + 100(40227)x + 1000(-8423) = 0$$

$$h(x) = x^3 + 4590x^2 + 4022700x - 8423000 = 0 \quad \text{(4)}$$

~~h(2) = negative~~ ✓ $\qquad \dfrac{8423000}{4022700} = 2.093$

~~h(3) = Positive~~ ✓

$\therefore$ The equation (4) has a root between 2 and 3

So we shall diminish the roots by 2

2	1	4590	4022700	-8423000
	0	2	9184	8063768
	1	4592	4031884	$\boxed{-359232}$
	0	2	9188	
	1	4594	$\boxed{4041072}$	
	0	2		
	1	$\boxed{4596}$		
	0			
	$\boxed{1}$			

The transformed equation is :

$$x^3 + 4596x^2 + 4041072x - 359232 = 0$$

we shall multiply the roots of this equation by 10, Then the transformed equation is

$$1(x^3) + 45960x^2 + 404107200x - 359232000 = 0 \quad \text{(5)}$$

$$\frac{359232000}{404107200} = 0.8889$$

$\therefore$ The equation (5) has a root between 0 and 1

From the equations (1) to (5) we diminished the roots by 1, 5, 3, 2 and found that one root of the equation (5) lies between 0 and 1.

Scanned by CamScanner

All these simply that the required root is

$1.5320\ldots = 1.532$ (By Correct to 3 decimal Places.

Example 2 Find the positive root of the equation
$x^3 - 2x^2 - 3x - 4 = 0$ Correct to 2 decimal Places.
using Horner's rule approximation.